

Master's Thesis Proposal:
Algorithms for Bounding Folkman Numbers

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Introduction

The study of Folkman numbers comes from the field of Ramsey theory. Ramsey theory is often phrased as the “party question:” What is the minimum number of people needed such that either r people all know each other or l people don’t know each other? Clearly, such statements can be phrased as graph coloring problems: What is the smallest 2-colored complete graph such that for all colorings there exists r vertices of the first color all of which are connected to one another (i.e., form a K_r subgraph) or l vertices of the second color all of which are not connected to one another (i.e., form an independent set).

The study of Folkman numbers is similar and has two parts. Either one can color the vertices of a graph or one can color the edges. This thesis will tackle both problems. In both cases, the problem is similar. Take vertex coloring. Given a vertex 2-colored graph, is it possible to find *for every coloring* a K_r in the first color or a K_l in the second color? For a given r, l the vertex Folkman number is the smallest satisfying graph. Furthermore, we restrict the possible graphs to those that don’t contain K_p . Also, instead of coloring the vertices we may color the edges. In this case, we are dealing with edge Folkman numbers.

Folkman number theory and Ramsey theory are pure mathematical areas, but we wish to bring them into the realm of computer science by applying computer algorithms to solve these problems.

This thesis has three parts. First, a study and discussion of Folkman properties. Although many papers have been written on the subject there is very little general overview and history available. Perhaps this thesis will be one of the first such overviews. The second part will tackle a specific vertex Folkman number. Work has already begun in this area and will hopefully produce results. Nenov showed in [21] that $10 \leq F_v(2, 2, 3; 4) \leq 14$ but the exact value is still unknown. This thesis aims to find the exact value. The third part concerns edge Folkman numbers. The specific edge Folkman number in question, $F_e(3, 3; 4)$, has been around since the 1960s and has proven very difficult. There is currently an upper bound, but that stands at 3×10^9 . There has already been some work done at reducing this using computers by Professor Radziszowski, but that research is still on-going. If the research is successful it would show that the upper bound is 127.

Folkman Numbers

First it will be necessary to define a few terms that will be important throughout this paper. Let G be a graph. The set of vertices in G is written $V(G)$ and the set of edges in G is written $E(G)$. The *order* of a graph is simply the number of vertices in G and is written $|V(G)|$. A K_n graph is a graph of order n such that all the vertices are connected to each other. This can also be called a *clique* of order n . The order of the largest clique in G is written $\text{cl}(G)$. The opposite of a clique is an *independent set*. This is where all the vertices in the subgraph are not connected to any other vertex.. The *maximum independent set* of G is the

independent set within G with the greatest order.

For positive r, l we say $G \rightarrow (r, l)^v$ if every 2-coloring of the vertices of G forces in G a monochromatic K_r subgraph in the first color or a monochromatic K_l subgraph in the second color. G is an element of the set $H(r, l; p)$ if and only if $G \rightarrow (r, l)^v$ and $\text{cl}(G) < p$, where $\text{cl}(G)$ is the maximum clique of G . The vertex Folkman number $F_v(r, l; p) = \min\{|V(G)| : G \in H(r, l; p)\}$. This notation can be generalized to n -colorings:

$H_v(a_1, a_2, \dots, a_k; p)$ – the set of all graphs G , $\text{cl}(G) < p$, such that for each vertex k -coloring of G there exists a K_{a_i} for $1 \leq i \leq k$ in color i .

$F_v(a_1, a_2, \dots, a_k; p) = \min\{|V(G)| : G \in H(a_1, a_2, \dots, a_k; p)\}$

For edge Folkman numbers the definitions are nearly the same:

$H_e(a_1, a_2, \dots, a_k; p)$ – the set of all graphs G , $\text{cl}(G) < p$, such that for each edge k -coloring of G there exists a K_{a_i} for $1 \leq i \leq k$ in color i .

$F_e(a_1, a_2, \dots, a_k; p) = \min\{|V(G)| : G \in H(a_1, a_2, \dots, a_k; p)\}$

Background

In 1970 Jon Folkman proved in [4] that there exists a graph G such that $G \rightarrow (a_1, \dots, a_k)$ and $\text{cl}(G) = \max\{a_1, \dots, a_k\}$. Thus, if $p > \max\{a_1, \dots, a_k\}$ then the Folkman number $F_v(a_1, \dots, a_k; p)$ exists. Folkman proposed the problem of finding the critical graphs at which point no smaller graph can satisfy the given vertex Folkman graph properties. That is, no smaller graph is in the specified set H . One important question is what makes these graphs critical and how can they be found without extensive computation or proof. Professor Nedyalko Nenov has written many papers [8, 9, 10, 17, 18, 19, 20, 21] on the subject, a few of which are important to this thesis.

Much of the current literature focuses on 2-colorings, as this is generally an easier problem to solve. Since 2-colored graphs can be represented as binary strings, the “color” of each vertex is either a 1 or a 0. This simplifies many things when writing software to study these graphs.

The graphs that are the current focus of my independent study, discussed below, are 3-colored. Research into these graphs has come from primarily two papers in which Nenov generalizes the bounds of Folkman numbers for many 3-colorings. Only in a few cases is he able to provide exact numbers. The papers will be discussed below.

Work Already Completed

The topic of an independent study conducted in Winter 2003/2004 was to look at the Folkman numbers $F_v(2, 2, 4; 5)$ and $F_v(2, 2, 3; 4)$. Nenov proved in [20] that $F_v(2, 2, 4; 5) = 13$. One area of the study was to verify his proof computationally.

This involved writing the program *h224_5* that could check a given graph to see if it satisfied the required properties for the Folkman graph. The program *geng*, part of the *nauty* package developed by Brendan McKay, generated all the non-isomorphic graphs of a given order. These graphs were then piped into *h224_5*. Graphs that are in the set $H(2, 2, 4; 5)$ are displayed.

Checking the upper bound $F_v(2, 2, 4; 5) \leq 13$ was simply a matter of supplying *h224_5* with the graph Q used by Nenov in his proof. Checking the lower bound is more difficult, and is still the subject of investigation. One possible technique would be to check all graphs of order ≤ 12 and determine that none are in the set $H(2, 2, 4; 5)$. However, this presents huge computational obstacles as the number of graphs grows considerably with the number of vertices. This number can hopefully be reduced by only generating graphs that do not contain K_5 . Currently, all graphs are generated and then checked for K_5 . This obviously takes time to do and is extraneous overhead. It would be better to take graphs on 10 vertices and extend them to graphs on 11 vertices without K_5 and then extend those to graphs on 12 vertices without K_5 .

Concerning $F_v(2, 2, 3; 4)$, Nenov showed in [21] that $10 \leq F_v(2, 2, 3; 4) \leq 14$. This is a specific case of his more general assertion that $2p + 4 \leq F_v(2, 2, p; p + 1) \leq 4p + 2$. The exact value of this Folkman number is still unknown. The current approach to this problem is similar as with $F(2, 2, 4; 5)$. The graph Γ supplied by Nenov showing an upper bound of 14 was run against the program *h223_4*. By using *geng* for graphs of order ≤ 11 the lower bound has been raised to 12. To attack the upper bound, different techniques will be needed as simply analyzing graphs of order 12 and 13 is, as stated before, computationally challenging.

In summary, the following areas have been explored:

- The programs *h224_5* and *h223_4* were written to check if graphs are in the sets $H(2, 2, 4; 5)$ and $H(2, 2, 3; 4)$, respectively.
- The lower bound on $F_v(2, 2, 3; 4)$ was raised to 12 and the upper bound of 14 was verified.
- A few techniques for more efficiently checking the higher bounds were explored and constitute future work.

Thesis Work

As stated before, the thesis will consist of three main parts. First, a study and discussion of Folkman number theory and the history of how it developed. There are many papers related to the theory, but there is very little information about the history. Much of the literature research will focus on developing an overview of the field.

The second part will extend the research done during the independent study as discussed in the previous section. There are many ideas as to how to approach finding the exact value for $F_v(2, 2, 3; 4)$. These ideas are discussed below.

Based on the techniques used in Nenov’s proofs, it may be possible to create algorithms directly from these techniques that could be extended to other problems. Further research is necessary to determine whether or not this is feasible. The hope is that work done in this area will help with finding the exact value for $F_v(2, 2, 3; 4)$ and also provide insight into solving more generic Folkman graph problems.

If it is the case that some of the techniques Nenov employs in his proofs can be used to create algorithms, then a specific set of library functions can be written that use these findings. This would clearly prove very useful in further research and possibly allow other Folkman numbers to be found more efficiently.

There has been some encouraging ideas that using the distributed processing power of the *condor* system will help solve $F_v(2, 2, 3; 4)$. The process of checking one graph is independent of checking another one. This means that the collection of graphs to be checked can be divided over many machines. Given enough machines, the time required to check all the graphs – even given the number of graphs that would have to be checked – can be considerably lessened.

Another method of reducing the upper bound of $F_v(2, 2, 3; 4)$ is to run heuristic searches over the graph space. By taking the existing graph Γ it may be possible to reduce it using various heuristic algorithms and find a new satisfying graph with 13 or even 12 vertices. If this is done, then with the new lower bounds discussed above, the exact value will be known. One possible heuristic would be simulated annealing. This technique is used to avoid local maximums. Since a “good fit” solution is not desired – this problem requires an exact answer – there needs to be a way to locate a better possible solution, rather than focus on one path that won’t lead to a solution; simulated annealing does just this.

Simulated annealing uses a random variable and a “cooling schedule” to determine how much a potential solution is allowed to change. It models the cooling process of substances. Initially, a lot of change is allowed – the substance is “hot.” Over time as the substance is cooled, the amount of random movement decreases. Eventually, the algorithm will settle on a solution. The randomness allows the algorithm to escape from local maximums.

The third part of the thesis will explore *edge* Folkman graphs, where the edges of graphs are colored instead of the vertices. The preceding work on vertex Folkman numbers will hopefully provide enough background to work on the edge Folkman number $F_e(3, 3; 4)$. The question of whether this has a reasonable bound has existed since the 1960s, but no one has been able to make significant progress because the search space is so enormous.

Professor Radziszowski has been working on reducing the upper bound for $F_e(3, 3; 4)$ which currently stands at 3×10^9 , as proven by Spencer and later corrected by [23]. The idea is to phrase the question as a satisfiability problem.

If a satisfying condition does *not* exist, then it would be known that $F_e(3, 3; 4) \leq 127$ – a substantial improvement over the current upper bound. The previous work in vertex Folkman numbers will hopefully prove useful, although new techniques and heuristics will be needed for dealing with graphs on the order of thousands or millions of vertices.

Deliverables

The main deliverable will be the results of the research into $F_e(3, 3; 4)$. This could potentially lead to a reduction on the upper bound, but as this problem has proven difficult in the past, it may be too much to foresee a solution for this thesis. In addition to the main results, it is hoped that an exact value for $F_v(2, 2, 3; 4)$ can be found. It is unclear at the moment which of the techniques discussed above will prove most successful, but research into each of these areas is important. As this research is conducted, it will be important to write software to perform experiments with different algorithms and heuristics. This software will also be considered a deliverable.

To review, the main points of this thesis will be:

- Study and understand vertex and edge Folkman number theory. Research its history and how it has developed through an exhaustive search for literature.
- Develop algorithms, software tools and libraries for attacking vertex and edge Folkman numbers.
- Attempt to find the exact value for $F_v(2, 2, 3; 4)$.
- Research the problems associated with attacking edge Folkman numbers of large graphs, with specific focus on an attempt to improve the upper bound of $F_e(3, 3; 4)$.

Schedule

Below is an estimated work schedule. Some of the dates are dependent on whether or not certain goals are reached so the work has been divided into two tracks.

March 15 Modify code to extend lower order graphs. Aim to raise the lower bound to 13. Work on verifying the accuracy of the current code and proving that it properly discards graphs.

March 22 Work on distributing calculations of $F_v(2, 2, 3; 4)$ across CS network using condor.

April 5 Finish literature search.

April 5 If lower bound for $F_v(2, 2, 3; 4)$ is raised to 13 begin *Track 1*. If exact value proven to be 12 begin *Track 2*. *Track 1*: Begin looking into heuristic techniques for lowering upper bound. *Track 2*: write up report on findings.

April 12 *Track 1*: Have working heuristics for lowering the upper bound of $F_v(2, 2, 3; 4)$. *Track 2*: Begin studying $F_e(3, 3; 4)$.

April 26 *Track 1*: Continue with heuristics. Work on *Track 2*. *Track 2*: Create a plan of attack for $F_e(3, 3; 4)$. Includes possible heuristics and algorithms to try.

May-July Implement heuristics and build a library of tools for solving specific problems.

September Begin putting together thesis paper.

November 1 Master's defense.

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