Master's Thesis Preproposal: Algorithms for Bounding Folkman Numbers

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Introduction

Let G be a graph. For positive r, l we say $G \to (r, l)^v$ if every two colouring of the vertices of G forces in G a monochromatic K_r subgraph in the first colour or a monochromatic K_l subgraph in the second colour. G is an element of the set $H(r, l; p)$ if and only if $G \to (r, l)^v$ and cl(G) ¡ p, where cl(G) is the maximum clique of G. The vertex Folkman number $F_v(r, l; p)$ is the order of the minimum ordered graph in $H(r, l; p)$. This notation can be generalised to n-colourings. $F_v(a_1, a_2, \ldots, a_n; p)$ is the order of the minimum ordered graph in $H(a_1, a_2, \ldots, a_n; p)$.

Previous Work

In the 1960s Jon Folkman proposed the problem of finding the critical graphs at which point no smaller graph can satisfy the given vertex Folkman graph properties. That is, no smaller graph is in the specified set H. One important question is what makes these graphs critical and how can they be found without extensive computation or proof. Professor Nedyalko Nenov has in more recent years written many papers on the subject, a few of which are important to this thesis.

Much of the current literature focuses on 2-colourings, as this is generally an easier problem to solve. The two graphs that are the current focus of my independent study, discussed below, are 3-coloured. Nenov has written several papers which generalise the bounds for a large number of 3-colourings, but only in a few cases is he able to provide exact numbers.

Current Work

The topic of my current independent study has been to look at the Folkman numbers $F_v(2, 2, 4; 5)$ and $F_v(2, 2, 3; 4)$. Nenov proved in [1] that $F_v(2, 2, 4; 5) = 13$. One area of my study has been to verify his proof computationally. This involved writing the program *h224_5* that could check a given graph to see if it satisfied the required properties for the Folkman graph. To generate the graphs, I used the program *geng*, part of the *nauty* package developed by Brendan McKay, which generates all non-isomorphic graphs of a given order. These graphs are then piped into *h224_5*. Graphs that are in the set $H(2, 2, 4; 5)$ are then displayed.

Checking the upper bound $F_v(2, 2, 4; 5) \leq 13$ was simply a matter of supplying *h224_5* with the graph Q used by Nenov in his proof. Checking the lower bound is more difficult, and is still the subject of investigation. One possible technique would be to check all graphs of order ≤ 12 and determine that none are in the set $H(2, 2, 4; 5)$. However, this presents huge computational obstacles as the number of graphs grows considerably with the number of vertices. This number can hopefully be reduced by only generating graphs that do not contain K_5. As of now, all graphs are generated and then checked for K_5. This obviously takes time to do and is extraneous overhead. It would be better to take graphs on 10 vertices and extend them to graphs on 11 vertices without K_5 and then extend those to graphs on 12 vertices without K_5.

Concerning $F_v(2, 2, 3; 4)$, Nenov showed in [2] that $10 \leq F_v(2, 2, 3; 4) \leq 14$. This is a specific case of his more general assertion that $2p + 4 \leq F_v(2, 2, p; p + 1) \leq 4p + 2$. The exact value of this Folkman number is still unknown. The current approach to this problem is similar as with $F(2, 2, 4; 5)$. The graph Γ supplied by Nenov showing an upper bound of 14 was run against the program *h223_4*. By using *geng* for graphs of order ≤ 11 the lower bound has been raised to 12. To attack the upper bound, different techniques will be needed as simply analysing graphs of order 12 and 13 is, as stated before, computationally challenging.

In summary, the following areas have been explored:

- The programs *h224_5* and *h223_4* were written to check if graphs are in the sets $H(2, 2, 4; 5)$ and $H(2, 2, 3; 4)$, respectively.
- The lower bound on $F_v(2, 2, 3; 4)$ was raised to 12 and the upper bound of 14 was verified.
- A few techniques for more efficiently checking the higher bounds were explored and constitute future work.

Future Work and Goals

The main goal for this thesis will be to explore a related field of study on *edge* Folkman graphs, where the edges of graphs are coloured instead of the vertices. Professor Radziszowski has been working on reducing the upper bound for $F_e(3, 3; 4)$ which currently stands at 3^8, as proven by Spencer [3]. First, however, I plan to finish the work on $F_v(2, 2, 3; 4)$ and continue to develop the software tools and libraries needed to tackle the more difficult project.

Based on the techniques used in Nenov's proofs, it may be possible to create algorithms directly from these techniques that could be extended to other problems. Further research is necessary to determine whether or not this is feasible. The hope is that work done in this area will help with finding the exact value for $F_v(2,2,3;4)$ and also provide insight to solving more generic Folkman graph problems.

Another method of reducing the upper bound of $F_v(2,2,3;4)$ is to run heuristic searches over the graph space. By taking the existing graph Γ it may be possible to reduce it using various heuristic algorithms and find a new satisfying graph with 13 or even 12 vertices. If this is done, then with the new lower bounds discussed above, the exact value will be known.

The preceding work will hopefully provide enough background to work on the edge Folkman number $F_e(3,3;4)$, which is the main goal for this thesis. The question of whether this has a reasonable bound has existed since the 1960s, but no one has been able to make significant progress because the search space is so enormous. Professor Radziszowski has found a way to turn the problem into a satisfiability problem which could yield promising results. If a satisfying condition can be found to his constraints, then it would be known that $F_e(3,3;4) \leq 127$ – a substantial improvement over the current upper bound 3^8. The previous work in vertex Folkman numbers will hopefully prove useful, although new techniques and heuristics will be needed for dealing with graphs on the order of thousands or millions of vertices.

To summarise, the main points of my research will be:

- Develop software tools and libraries for attacking vertex and edge Folkman numbers.
- Find the exact value for $F_v(2,2,3;4)$.
- Research the problems associated with attacking edge Folkman numbers of large graphs, with specific focus on improving the upper bound of $F_e(3,3;4)$.

References

[1] Nenov, N., *On the 3-Colouring Vertex Folkman Number* $F(2,2,4)$. Serdica Math. J. (2001), 131-136.

[2] Nenov, N., *On a Class of Vertex Folkman Graphs.* (2000).

[3] Spencer, J., *Three Hundred Million Points Suffice.* J Combin Theory Ser A 49 (1988), 210-217.