A Shorter Method for Calculating $n!$

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Theorem 1 *$n!$ requires only $\lceil \frac{n}{2}$ multiplications.*

Proof The proof requires two parts: one for n even, the other for n odd. We begin with n even.

For some n, which we'll pick to be greater than 3,

$$n! = (1)(2)\cdots(n-1)(n)$$

Now pair up the factors in the following manner:

$$n! = (1\cdot n)(2\cdot(n-1))\cdots$$

And distribute:

$$n! = n(2n-2)\cdots$$

For n even there are clearly only $\frac{n}{2}$ factors in this last expression. However, there is no advantage unless there is an easy way to generate the factors. Choose n to be sufficiently large that we can write the following:

$$\begin{array}{rcl} n! & = & (1\cdot n)(2\cdot(n-1))(3\cdot(n-2))(4\cdot(n-3))(5\cdot(n-4))(6\cdot(n-5))\cdots \\ n! & = & n(2n-2)(3n-6)(4n-12)(5n-20)(6n-30)\cdots \end{array}$$

In general, the k^{th} factor is $k(n-(k-1)) = kn - k(k-1), 1 \le k \le n$. The difference between two consecutive factors is then:

$$\begin{array}{c} [(k+1)n - (k+1)((k+1)-1)] - [kn - k(k-1)] \\ \left[(k+1)n - (k+1)^2 + (k+1)\right] - kn + k^2 - k \\ kn + n - (k^2+2k+1) + (k+1) - kn + k^2 - k \\ kn + n - k^2 - 2k - 1 + k + 1 - kn + k^2 - k \\ n - 2k \end{array}$$

Now we define $n!$ in terms of a new function δ, $n! = \delta(n,n)$, where δ is:

$$\delta(x,y) = \begin{cases} 1 & y < 2 \\ x\cdot\delta(x+(y-2), y-2) & y \ge 2 \end{cases}$$

It can be seen from the very definition of δ that there will only be $\frac{n}{2}$ factors since each iteration decrements y by 2 causing it to reach 0 in $\frac{n}{2}$ steps. Once

$y < 2$ the recursion can stop since proceeding any further will not change the product.

We should stop here to justify the definition of δ. The single factor x is the k^{th} factor. $x + (y-2)$ is the $(k+1)^{\text{th}}$ factor (provided $(y-2) \geq 2$). The difference should be $n - 2k, k > 0$. So for the k^{th} iteration $(y-2)$ should equal $n - 2k$. We initially call δ with $y = n$. Thus on the k^{th} iteration $y - 2$ expands to

$$\begin{aligned}(\cdots((((n-2)-2)-2)-2)-2)\cdots-2 &= n\underbrace{-2-2-2-2-2\cdots-2}_{-2k} \\ &= n - 2k\end{aligned}$$

Now we need to handle the case when n is odd. Having shown the proof for n even, the proof for n odd is trivial. $n! = n \cdot (n-1)!$. If n is odd, then $n-1$ is even and so the complete definition for $n!$ is:

$$n! = \begin{cases} \delta(n,n) & n > 0 \text{ and even} \\ n(n-1)! & n > 0 \text{ and odd} \end{cases}$$