

# Computing the Vertex Folkman Number $F_v(2, 2, 3; 4)$

*Soon to be the Basis of CS Master's Thesis*

Jonathan Coles

Advisor: Professor Stanisław P. Radziszowski

`{jpc1870,spr}@cs.rit.edu`

Department of Computer Science, Rochester Institute of Technology



# Overview



- Notation
- What Are Folkman Numbers?
- Early Results
- Computing  $F_v(2, 2, 3; 4)$
- Results
- Questions



# Notation

All graphs are simple and undirected.

$\overline{G}$  – complement of graph  $G$

$V(G)$  – vertex set of graph  $G$

$E(G)$  – edge set of graph  $G$

$K_n$  – complete graph on  $n$  vertices

$\chi(G)$  – chromatic number of  $G$

$\alpha(G)$  – order of maximum independent set in  $G$

# What Are Folkman Numbers?

We write

$$G \rightarrow (a_1, \dots, a_k; p)^v$$

iff for every vertex  $k$ -coloring of  $G$ ,  $K_p \not\subseteq G$ , there exists a monochromatic  $K_{a_i}$  in color  $i$  for some  $i \in \{1, \dots, k\}$ .

$G$  is called a Folkman graph.

# What Are Folkman Numbers?

Alternatively, if

$$V(G) = V_1 \cup \dots \cup V_k, \quad V_i \cap V_j \neq \emptyset, i \neq j$$

is a partitioning of  $G$ , then  $K_{a_i} \subseteq G[V_i]$  for some  $i \in \{1, \dots, k\}$ .

# What Are Folkman Numbers?

The vertex Folkman number is defined as

$$F_v(a_1, \dots, a_k; p) = \min\{|V(G)| : G \rightarrow (a_1, \dots, a_k; p)^v\},$$

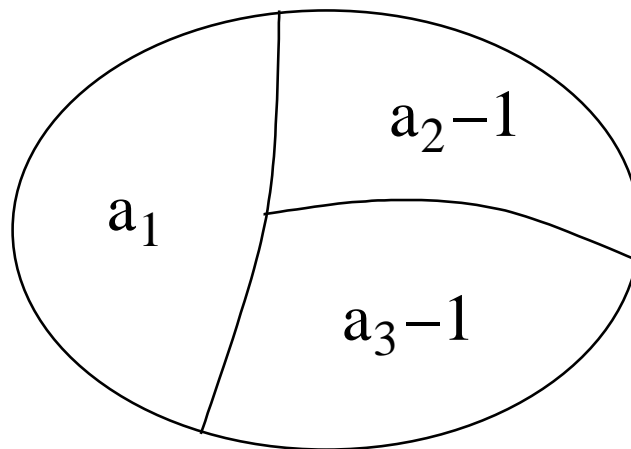
the least number of vertices required for there to exist a Folkman graph with the specified parameters.

# Early Results

Let  $m = 1 + \sum_1^k (a_i - 1)$  then

$$F_v(a_1, \dots, a_k; p) = m, \quad p > m$$

Take  $K_m$  and divide it into  $k$  parts. By the pigeon-hole principle, one part must contain a  $K_{a_i}$  in part  $i$  for some  $i \in \{1, \dots, k\}$ .



# Early Results



In 2000, a much harder proof by Łuczak, Ruciński, and Urbański showed:

$$F_v(a_1, \dots, a_k; p) = a_k + m, \quad p = m$$

What about  $p < m$ ?



# Early Results

For  $p = m - 1$ , there have been many results, but no grand generalizations.

E.g., for any  $r \geq 3$ ,

$$F(\underbrace{2, \dots, 2}_r; r) = \begin{cases} 11 & r = 3 \text{ or } r = 4 \\ r + 5 & r \geq 5 \end{cases}$$

For  $p = m - 2$ , very little has been done. Jenson and Royle (1995) used a computer search to find

$$F_v(2, 2, 2, 2; 3) = 22$$

# Computing $F_v(2, 2, 3; 4)$

## Motivation:

- Understand the difficulty in computing Folkman numbers
- Develop algorithms for finding Folkman numbers
- Contribute new results to the field

# Computing $F_v(2, 2, 3; 4)$

In 2000, Nenov proved

$$2q + 4 \leq F_v(2, 2, q; q + 1) \leq 4q + 2$$

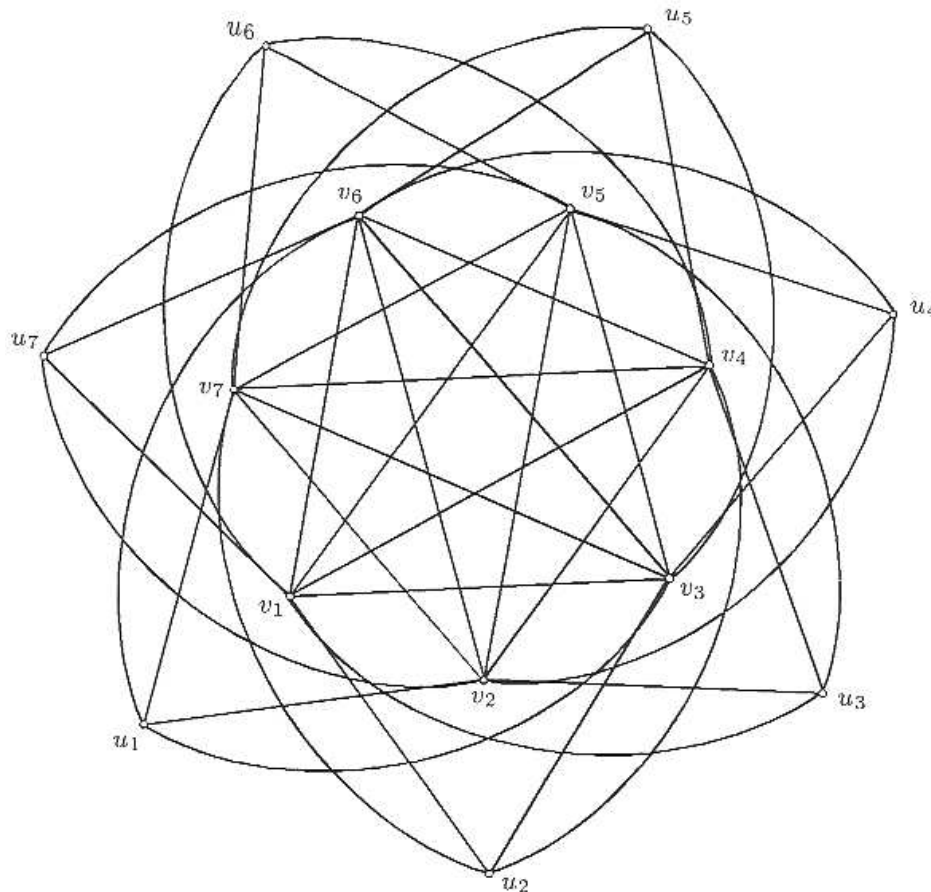
For  $q = 3$  ( $m = 5$ ),

$$10 \leq F_v(2, 2, 3; 4) \leq 14$$

So the problem is to find the exact value.

# Computing $F_v(2, 2, 3; 4)$

$\Gamma_3$  (Nenov) proves the upper bound is 14:



# Computing $F_v(2, 2, 3; 4)$

Possible approaches:

- Search all graphs of order 10, 11, 12, 13 until a Folkman graph is found
- Find all Folkman graphs on 14 vertices and drop vertices until the graphs are no longer Folkman

# Computing $F_v(2, 2, 3; 4)$

$ V(G) $	# of Graphs	Search Time (1 GHz CPU)
8	12346	0.97 seconds
9	274668	17.51 seconds
10	12005168	~9 minutes
11	1018997864	~14 hours
12	165091172592	~96 days
13	50502031367952	~80 years
14	29054155657235488	~46,000 years

Brute force searching is out.

How to generate all Folkman graphs on 14 vertices?

# Computing $F_v(2, 2, 3; 4)$

A result from Ramsey theory states that  $R(4, 3) = 9$ .  
Thus, all graphs  $|V(G)| \geq 9$  have either  $K_4$  or  $\overline{K_3}$ .

The Folkman graphs we are interested in do not have  $K_4$ ,  
so they must have an independent set of order 3.

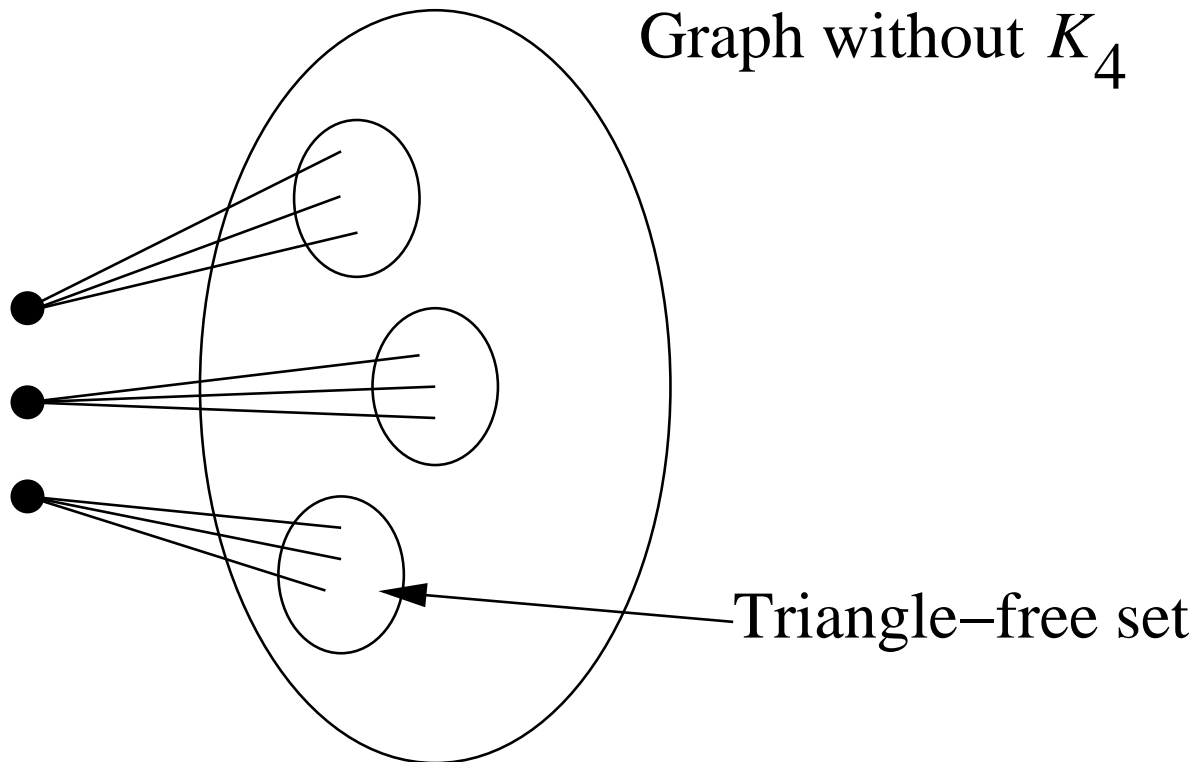
This information can be used to generate graphs on 14  
vertices by extending  $K_4$ -free graphs on 11 vertices by  
adding  $\overline{K_3}$ .

# Computing $F_v(2, 2, 3; 4)$

To find all Folkman graphs on 14 vertices:

1. Find all  $K_4$ -free graphs on 11 vertices [easy]
2. Extend to 14 vertices by adding (efficiently!)  $\overline{K_3}$  [hard]
3. Filter out isomorphs (*nauty* by McKay) [easy]
4. Filter for Folkman property [easy]

# Computing $F_v(2, 2, 3; 4)$



# Results

$\chi(G)$	$E(G)$	$\max(\deg(G))$	$\min(\deg(G))$	$\alpha(G)$					
#Graphs	#Graphs	#Graphs	#Graphs	#Graphs					
5	12227	42	1	7	527	4	451	3	1507
		43	6	8	11080	5	5759	4	10557
		44	51	9	393	6	5996	5	160
		45	453	10	227	7	21	6	2
		46	2279					7	1
		47	4555						
		48	3628						
		49	1138						
		50	114						
		51	2						

# Results



After dropping one vertex in all possible ways for each Folkman graph, no resulting graph was Folkman.

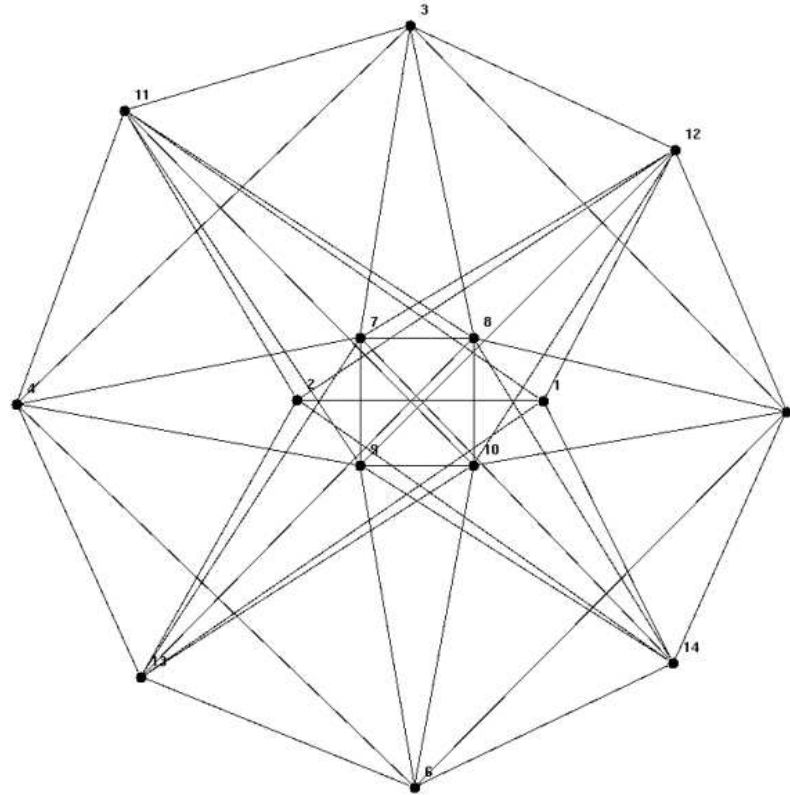
Thus,

$$F_v(2, 2, 3; 4) = 14.$$



# Results

We found the most symmetric Folkman graph on 14 vertices (16 automorphisms):



# Verification



To verify the results, a similar extension was performed on graphs on 10 vertices.

The problem was slightly different as one must add 4 vertices, not 3. Other results from Ramsey theory aided with this.

The graphs from the second extension technique matched those from the first, thus the results were verified to be correct.



# Questions

---



Thank you very much.

<http://www.jpcoles.com/uni/rit/thesis/>

Questions?

